Mathematics and Politics

The Quantification of Power

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Weighted voting

Suppose that there are 7 voters for a particular election, voting between three candidates A , B , C .

1. Unanimous: winner needs all votes A A A A A A A
2. Majority: winner needs 4 votes A A A A B B C
3. Quota: winner needs 5 votes A A A A A B C

Weighted voting

How can we numerically represent these particular voting schemes?

1. Unanimous: $V(7||1,1,1,1,1,1,1)$ A A A A A A A
2. Majority: $V(4||1,1,1,1,1,1,1)$ A A A A B B C
3. Quota: $V(5||1,1,1,1,1,1,1)$ A A A A A B C

Weighted voting

Example

United Nations Security Council has

- 5 permanent members (U.S., China, England, France, Russia).
- 10 other countries that rotate (currently Belgium, Dominican Republic, Estonia, Germany, Indonesia, Niger, Saint Vincent and the Grenadines, South Africa, Tunisia, Vietnam).

Weighted voting

Example

- A measure passes if 9 members vote for it.
- Any of the 5 permanent members have veto power.
- Is this a weighted voting system?
- Yes! It can be represented as

$$V(49||9,9,9,9,9,1,1,1,1,1,1,1,1,1,1)$$

- Or as

$$V(39||7,7,7,7,7,1,1,1,1,1,1,1,1,1,1)$$

- We say these systems are *isomorphic*.

Example

Any two unanimity methods are isomorphic because all votes are required, regardless of their weights.

Weighted voting

Scary example

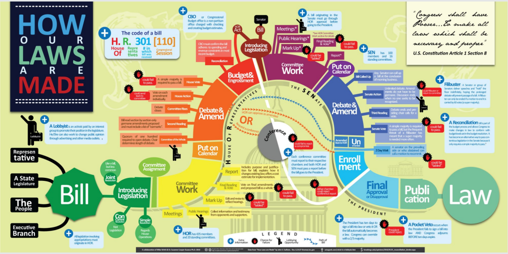
- In the Electoral College, each state has electoral votes that count toward the presidential election.
- We can think of this as having 51 voters with different weights.
- So California is a voter with weight 55, and Massachusetts is a voter with weight 11.
- For a candidate to win the election, 270 votes are needed.
- This can be encoded with the notation

$$V(270||55, 38, 29, 29, 20, 20, 18, 16, ..., 4, 3, 3, 3, 3, 3, 3, 3, 3)$$

Weighted voting

Example

Passing laws in the U.S. is a complicated thing...



In short, a proposed legislation has to have the support of

- a majority of the House,
- a majority of the Senate,
- and the President.
- If the President does not support it, he can veto it...

Weighted voting

Example

- ... but if the legislation has the support of 2/3 of the House and the Senate, then he cannot veto it and the legislation passes.
- Vice-president plays a role since he can break ties in the Senate.
- Is this a weighted voting system?

Proposition

No.

Let's look at a couple of interesting examples...

Weighted voting

Example

- Suppose a parliament has representatives from three parties:
  - A with 49 members,
  - B with 49 members,
  - D with 2 members.
- Simple majority (≥ 51 votes) wins the vote.
- So this is a $V(51||49, 49, 2)$ weighted voting system.
- **But D is not disadvantaged!**
- In fact $V(51||49, 49, 2)$ and $V(2||1, 1, 1)$ are isomorphic.

Weighted voting

Example

Something like the previous example happened in the Senate in 2001:

- In 2001, U.S. Senate had 50 Republicans and 50 Democrats.
- Jim Jeffords, a Republican, became an independent in 2001.
- Vice-president was a Republican, so the division in the Senate became 50-50-1.

Jim Jeffords
(1934-2014)

- In the scheme $V(51||50, 50, 1)$, neither the Republicans nor the Democrats could accomplish anything unless Jeffords joined them.
- The scheme is actually isomorphic to $V(2||1, 1, 1)$.

Weighted voting

Example

- In the scheme $V(51||50, 50, 1)$, voter D has only one vote but is just as important as A and B.
- Suppose a parliament has representatives from four parties:
  - A with 26 members,
  - B with 26 members,
  - C with 26 members,
  - D with 22 members.
- Simple majority $V(51||26, 26, 26, 22)$ will win.
- What is the situation with D ?
- $\{A\}$ and $\{A, D\}$ yield the same outcome. $\{A, B, D\}$ and $\{A, B\}$ yield the same outcome.
- **Voter D has nothing!**

Weighted voting

How much power does each voter have?

- In $V(51||50, 50, 1)$, voter D had less than 1% of the votes, but it had the same influence as the other parties. **You cannot say that A has 50% more power than D.**
- In $V(51||26, 26, 26, 22)$, voter D had 22% of the votes and had no influence at all.
- Can we somehow quantify this influence, or voting power?
- Yes! Using *power indices*.

Winning and Losing Coalitions

- Consider $V(51||26, 26, 26, 22)$.
- A with 26 votes,
- B with 26 votes,
- C with 26 votes,
- D with 22 votes.

If A and B banded together, they would have 52 votes, and would decide the election. Then $\{A, B\}$ is called a *winning coalition*. Similarly $\{A, B, D\}$ is a winning coalition, but $\{C, D\}$ is a losing coalition.

- A *coalition* is any subset of $\{A, B, C, D\}$.
- A coalition is a *winning coalition* if, when everyone in it votes for a candidate, that candidate wins.
- A coalition that is not winning is a *losing coalition*.

Critical Voters

Example

- Suppose there are four voters: A ,B ,C ,D and the voting scheme is

$$V(14||10^A, 8^B, 5^C, 2^D)$$

- For example, the coalition $\{A, B, D\}$ is winning since the total of their votes is $10 + 8 + 2 = 20 \geq 14$.
- The coalition $\{C, D\}$ is not winning since the total of their votes is $5 + 2 = 7 < 14$.
- We say that a voter V in a winning coalition is *critical* if its remove results in a losing coalition.
- For $\{A, B, D\}$, voters A and B are critical but D is not.

Computing the Banzhaf power index

Example

$$V(14||10^A, 8^B, 5^C, 2^D)$$

Coalition	# of votes	Winning?	Critical voters
$\{\ \}$	0	No	n/a
$\{A\}$	10	No	n/a
$\{B\}$	8	No	n/a
$\{C\}$	5	No	n/a
$\{D\}$	2	No	n/a
$\{A, B\}$	18	**Yes**	A ,B
$\{A, C\}$	15	**Yes**	A ,C
$\{A, D\}$	12	No	n/a
$\{B, C\}$	13	No	n/a
$\{B, D\}$	10	No	n/a
$\{C, D\}$	7	No	n/a
$\{A, B, C\}$	23	**Yes**	A
$\{A, B, D\}$	20	**Yes**	A ,B
$\{A, C, D\}$	17	**Yes**	A ,C
$\{B, C, D\}$	15	**Yes**	B ,C ,D
$\{A, B, C, D\}$	25	**Yes**	None

Computing the Banzhaf power index

Example

- There are 12 instances when a voter is critical in $V(14||10^A, 8^B, 5^C, 2^D)$.
- Let $c(P)$ be the number of times that P is critical.
- $c(A) = 5$, $c(B) = 3$, $c(C) = 3$, $c(D) = 1$. So

$$\text{Power index of A} = \frac{5}{12} = 0.42 = 42\%$$
$$\text{Power index of B} = \frac{3}{12} = 0.25 = 25\%$$
$$\text{Power index of C} = \frac{3}{12} = 0.25 = 25\%$$
$$\text{Power index of D} = \frac{1}{12} = 0.08 = 8\%$$

- Note that A has only two more votes than B, but is much more powerful.
- Even though B and C have different number of votes, they have the same power.

Banzhaf Power Index

Computing the power index

- Then the *Banzhaf power index* of a voter V is

$$\frac{\text{number of times voter } V \text{ is critical across all coalitions}}{\text{total number of times all voters are critical}}$$

- This really computes the **probability** that a voter V will change the outcome of a vote if they join a coalition.

Back to the example of $V(51||50^{A}, 50^{B}, 1^{D})$:

Winning coalitions: $\{A, B\}$, $\{A, D\}$, $\{B, D\}$, $\{A, B, D\}$. There are 6 critical cases, and each voter appears twice. So each has a power index of $2/6 = 33.3\%$.

In $V(51||26, 26, 26, 22)$, voter D has power index 0%.

Example: European Economic Community

European Economic Community of 1958 (future EU)

- European Economic Community consisted of 6 countries:
  - France with 4 votes
  - Germany with 4 votes
  - Italy with 4 votes
  - Belgium with 2 votes
  - Netherlands with 2 votes
  - Luxembourg with 1 vote
- 12 votes are needed to win (quota).
- So we have a $V(12||4,4,4,2,2,1)$ weighted voting system.

European Economic Community

European Economic Community of 1958 (future EU)

Here are the winning coalitions:

Coalition	# of votes	Critical voters
$\{F, G, I\}$	12	F,G,I
$\{F, G, I, B\}$	14	F,G,I
$\{F, G, I, N\}$	14	F,G,I
$\{F, G, I, L\}$	13	F,G,I
$\{F, G, B, N\}$	12	F,G,B,N
$\{F, I, B, N\}$	12	F,I,B,N
$\{G, I, B, N\}$	12	G,I,B,N
$\{F, G, I, B, N\}$	16	none
$\{F, G, I, B, L\}$	15	F,G,I
$\{F, G, I, N, L\}$	15	F,G,I
$\{F, G, B, N, L\}$	13	F,G,B,N
$\{F, I, B, N, L\}$	13	F,I,B,N
$\{G, I, B, N, L\}$	13	G,I,B,N
$\{F, G, I, B, N, L\}$	17	none

European Economic Community

European Economic Community of 1958 (future EU)

$$V(12||4,4,4,2,2,1)$$

- There are 42 instances when a voter is critical.
- France, Germany, and Italy are critical 10 times, Belgium and Netherlands 6 times, Luxembourg zero times. So

$$\begin{aligned}
\text{Power index of France} &= 10/42 = 0.24 = 24\% \\
\text{Power index of Germany} &= 10/42 = 0.24 = 24\% \\
\text{Power index of Italy} &= 10/42 = 0.24 = 24\% \\
\text{Power index of Belgium} &= 6/42 = 0.14 = 14\% \\
\text{Power index of Netherlands} &= 6/42 = 0.14 = 14\% \\
\text{Power index of Luxembourg} &= 0/42 = 0.00 = 0\%
\end{aligned}$$

- Luxembourg has no power even though it has a vote; it is a *dummy* voter.

Example: UN Security Council

Banzhaf power index of the members of the
United Nations Security Council

The Council has

- 5 permanent members (U.S., China, England, France, Russia).
- 10 other countries that rotate (currently Belgium, Dominican Republic, Estonia, Germany, Indonesia, Niger, Saint Vincent and the Grenadines, South Africa, Tunisia, Vietnam).

UN Security Council

- A measure passes if 9 members vote for it.
- Any of the 5 permanent members have veto power.
- This is a weighted voting system

$$V(39||7,7,7,7,7,1,1,1,1,1,1,1,1,1,1)$$

  (this is one way of representing it).
- This makes sense since:
  - If five permanent members vote yes, this gives 35 votes, so to clear the quota of 39, four more are needed.This means 9 countries voted yes, as required.
  - If any of permanent members does not vote yes, then the most votes that can be gathered is $4 \cdot 7 + 10 \cdot 1 = 38$. This means that permanent members have veto power.

Banzhaf index of the UN Security Council

- Any winning coalition must have all five permanent members in it and at least four more non-permanent members.

$$\{\text{US, UK, Ch, Ru, Fr, DR, SVG, Es, Tun}\}$$

- So the winning coalitions looks like

$$\{5 \text{ permanent}, 4 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 5 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 6 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 7 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 8 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 9 \text{ non-permanent}\}$$
$$\{5 \text{ permanent}, 10 \text{ non-permanent}\}$$

- There are

$$210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$$

winning coalitions.

Banzhaf index of the UN Security Council

Who is critical in $V(39||7,7,7,7,7,1,1,1,1,1,1,1,1,1,1)$?

$$\{\text{US, UK, Ch, Ru, Fr, DR, SVG, Es, Tun}\}$$
$$\{\text{US, UK, Ch, Ru, Fr, Ge, DR, SVG, Es, Tun}\}$$

- Thus
  - Each of the five permanent members is critical 848 times, and
  - Each of the ten non-permanent members is critical 84 times.
- So the total number of times that some voter is critical is

$$5 \cdot 848 + 10 \cdot 84 = 5080$$

$$\text{Banzhaf index of each permanent member} = \frac{848}{5080} = 0.1669 = 16.69\%$$
$$\text{Banzhaf index of each non-permanent member} = \frac{84}{5080} = 0.165 = 1.65\%$$

Permanent members have about 10 times as much power!